When AC Power ≠ V A cos (Φ)

Abstract

Where a unit of measurement may be derived from different mathematical models, it is important to consider the suitability of any selected model to the nature of signal being measured. A legitimate equation applied in an environment for which that equation is not suited, will inevitably produce an incorrect result.
This document looks at the mathematical principal behind commonly used terms in power electronic measurement and considers where these equations are correctly applied.

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1 INTRODUCTION

Deriving models and equations to quantify and analyse systems is an essential component of engineering. The potentially overwhelming complexity of real-world systems are managed by applying constraints and approximations to produce simplified models. While this provides a useable tool, it is important to remain mindful of the error that simplified models may inevitably introduce.

In power electronics many equations are available for calculating power in different scenarios. Of which, one of the most common is:

$$Power = V_{rms} \times A_{rms} \times \cos(\Phi)$$

This equation is not *wrong* when used appropriately but it will produce the *wrong* answer; if as is often the case, it is assumed you can always calculate the true power in an electronic system with two Root Mean Square (RMS) measurement devices and a simple phase detector.

The error does not originate from the derivation of the equation, but in the fact that the assumptions upon which it is derived do not apply to many non-ideal, real-world devices. This document revisits some of the most common equations, their assumptions and how this can affect real measurements.

2 EQUATIONS FOR POWER

The starting point in calculating power is to average the energy over time. For steady-state Direct Current (DC) systems this can be any duration and for Alternating Current (AC) this is usually a single cycle.

$$Power = \frac{Energy}{Time} = \frac{1}{T}\int_0^T v(t) \times i(t)dt$$

The following sub-sections demonstrate how assumptions are applied to arrive at the common formula.

2.1 Pure Resistive Load

With a purely resistive load there is a fixed ratio, represented by the resistance between the instantaneous current and voltage.

$$v(t) = R \times i(t)$$

This equation can then be substituted into:

$$Power = \frac{1}{RT}\int_0^T v^2(t)\,dt \quad and \quad Power = \frac{R}{T}\int_0^T i^2(t)\,dt$$

Which can be rearranged to remove R:

$$Power = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}\ \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt}$$

Which, by inspection can be seen to be equivalent to

$$Power = V_{rms} \times A_{rms}$$

The formula can be used to calculate the power for distorted waveforms, but is limited to systems that are purely resistive, which is not applicable to the majority of systems.

2.2 Linear AC Load

In the analysis of AC circuits, it is common to assume a linear load, i.e., a load comprising a network of ideal resistors, capacitors, and inductors. In response to a single frequency component, voltage and current will be pure sine waves with the relative magnitude and phase angles dictated by the impedance. Therefore, voltage and current will be of the form:

$$v(t) = \mathrm{V}\sin(\omega t)$$

$$i(t) = A\sin\,(\omega t + \phi)$$

Substituting these values into the integral for power

$$Power = \frac{1}{T}\int_0^T v(t) \times i(t)dt = \frac{VA}{T}\int_0^T \sin(\omega t) \times \sin\,(\omega t + \phi)dt$$

This can be integrated and then arranged to give power in terms of the RMS voltage and current.

$$Power = \frac{VA}{\sqrt{2}\,\sqrt{2}}\cos(\phi) = V_{rms}\,A_{rms}\cos(\phi)$$

The $\cos(\phi)$ term is often referred to as the displacement power factor.

2.3 Current Harmonics with Non-Linear Loads

Recognising that in most real-world applications a load is unlikely to be linear, either because of non-ideal components or an active device, the current measured in response to an ideal sinusoidal voltage input will be distorted. If periodic, then the waveforms can be separated into a DC component and series of harmonics, as shown below.

$$v(t) = \mathrm{V}\sin(\omega t)$$

$$i(t) = A_0 + \sum_{n=1}^{N} A_n \sin(\omega nt + \phi_n)$$

Given the assumption that the voltage remains an ideal sine wave, the only power in this system will be in the fundamental harmonic and the power simplifies to:

$$Power = V_{rms} \times A_{1\_rms} \times \cos(\phi)$$

This is similar in form to the equation in section 2.2 but in this case, only the first (fundamental) harmonic component of the total RMS contributes to real power.

Alternatively, the harmonically distorted nature of the current may be presented using the definition of Total Harmonic Distortion (THD) of the current:

$$THD_i = \frac{\sqrt{\sum_{n=2}^{N} A_{n_{rms}}^2}}{A_1}$$

From which power may be defined as:

$$Power = V_{rms} \times A_{rms} \times \cos(\phi) \times \frac{1}{\sqrt{1 + THD_i^2}}$$

Again, this is of same form as section 2.2, but with a new term that is often referred to as the 'distortion power factor'.

In many applications, despite the voltage not being a pure sine wave, this formula continues to be used as a reasonable approximation. Voltage distortion is typically low and therefore a relatively small percentage of power is consumed in these harmonics.

However, when higher power accuracy is required, this approximation that ignores the higher order power harmonics should not be used because the power in those harmonics may equal or exceed the precision of the measurement.

2.4 INSTANTANEOUS SAMPLE MULTIPLICATION METHOD

In addition to the non-linear load and non-ideal voltage source, there are likely to be noise and other inter-harmonic components that are not included in the harmonics.

In order to obtain the closest measurement of true power, the original power integral can be represented by discrete samples of a cycle;

$$Power = \frac{1}{T}\int_0^T v(t) \times i(t)dt \approx \frac{1}{N}\sum_{n=1}^{N} v(n) \times i(n)$$

RMS values for voltage and current are obtained in a similar manner.

$$V_{rms} \approx \sqrt{\frac{1}{N}\sum_{n=1}^{N} v(n)^2} \qquad and \qquad A_{rms} \approx \sqrt{\frac{1}{N}\sum_{n=1}^{N} i(n)^2}$$

From these instantaneous sample measurements, the true power factor can be derived using the formula:

$$PF = \frac{power}{V_{rms}A_{rms}}$$

We use the term “true” power factor, because it reflects an absolute definition of the ratio between energy converted (W) and total available energy (VA), or what is sometimes referred to as the ratio between *real power* and *apparent power*.

2.5 HIERARCHY OF POWER FACTOR EQUATIONS

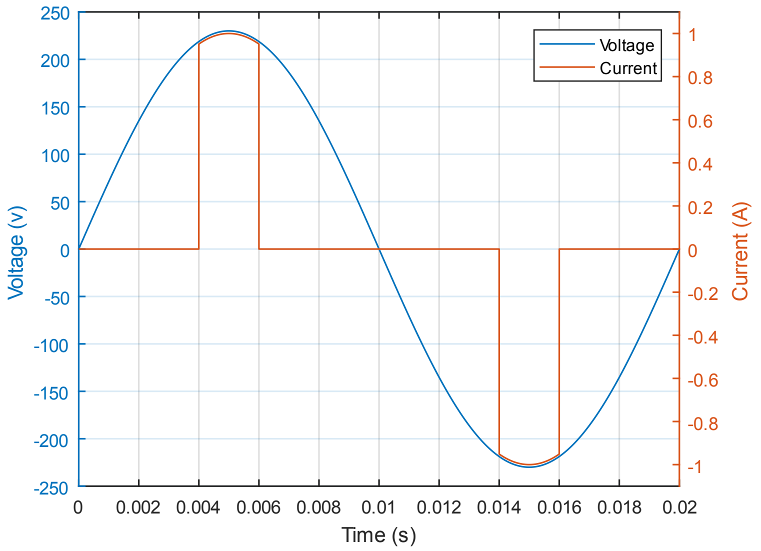
From the proceeding sub-sections, suitability of the three methods to derive power factor under varying conditions can be defined as follows:

	Suitable for:
Displacement Power Factor where $\mathrm{PF} = \cos(\phi)$	Phase displaced Sinewaves
Distortion Power Factor where $\mathrm{PF} = \cos(\phi) \times \frac{1}{\sqrt{1+THD_i^2}}$	Phase displaced and/or harmonically distorted current waveshapes
True Power Factor where $\mathrm{PF} = \frac{power}{V_{rms}A_{rms}}$	All waveshapes

3 AN EXAMPLE – PULSED CURRENT

To demonstrate the difference in varying power equations and the potential scale of the resultant errors, consider a Device Under Test (DUT) that draws power in short bursts; a common example of this type of behaviour is a Switched-mode power supply (SMPS). Typically, these designs consume power for only a small fraction of the cycle, when input capacitor charging current is drawn near the peak of the voltage input, which results in a current waveform that is highly distorted.

In this example an idealised resistor is switched in for part of the cycle; The supplied voltage is a 230 volt, 50Hz sine wave and the current draw occurs 1ms either side of the sine peak.



3.1 ANALYTIC SOLUTION

Integrating the instantaneous power and averaging gives the power of the DUT. Noting the symmetry in the waveform the problem is reduced by integration of a half-cycle, and power exists only where current is flowing:

$$Power = \frac{1}{T}\int_0^T v(t) \times i(t)dt = \frac{2VA}{T}\int_a^b \sin(\omega t) \times \sin(\omega t)dt$$

$$Power = \frac{VA}{T}\left[t - \frac{\sin(2\omega t)}{2\omega}\right]_a^b$$

Where $a = 0.004$ and $b = 0.006$ (the on and off time of the current switch), and the other values can be substituted to obtain a true power

$$Power = \frac{230 \times 1}{0.02}\left[t - \frac{\sin(4\pi 50\, t)}{4\pi 50}\right]_{0.004}^{0.006} \approx 44.516\, W$$

Next, calculate the RMS for both the voltage and current.

$$A_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\, dt} = \sqrt{\frac{2A}{T}\int_a^b \sin^2(\omega t)\, dt} = \sqrt{\frac{1A}{T}\left[t - \frac{\sin(2\omega t)}{2\omega}\right]_a^b}$$

$$A_{rms} = \sqrt{\frac{1}{0.02}\left[t - \frac{\sin(2\omega t)}{2\omega}\right]_{0.004}^{0.006}} = 439.94\, mA$$

$$V_{rms} = \frac{V}{\sqrt{2}} = \frac{230}{\sqrt{2}} = 162.63\, V$$

Using the true power and the RMS values, the true power factor (PF) is:

$$PF = \frac{true\ power}{apparent\ power} = \frac{44.516\, W}{162.63\, V \times 439.94\, mA} = 0.62219$$

This gives a theoretical displacement angle of:

$$Displacement\ Angle = \cos^{-1}(0.62219) = 51.524°$$

This angle clearly has no direct correlation to the waveform, so it would not be possible to obtain the correct Power Factor or the correct Power by calculation from voltage and current RMS measurements and simple phase detection.

4 WHICH METHOD SHOULD I USE?

In reviewing commonly observed power equations, it is evident that they are all simplifications of general theory, obtained by making assumptions about the load and associated waveforms.

Given this reality, it is always best to use a sampling and measurement technique that is as close as possible to the original instantaneous sampling equation, since this will involve the fewest assumptions and therefore result in proper measurement for the greatest number of applications.

In specific cases, such as the analysis of triplen harmonics in a three-phase system, it can then be a conscious decision to apply an associated model which may easily be applied if the measurement system is designed for instantaneous sampling.